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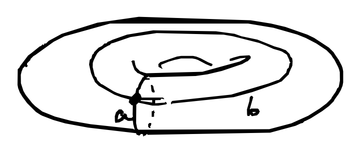
SOME EXAMPLES OF FUNDAMENTAL GROUPS

Theorem: $\pi_1(X\times Y, x_0\times y_0)$ is isomorphic to $\pi_1(X, x_0)\times \pi_1(Y, y_0)$.

Corollary: The fundamental group of the torus $S^1\times S^1$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ (with any base point)



Proof of Theorem: From the projections $\pi_X : X\times Y \longrightarrow X$
$\pi_Y : X\times Y \longrightarrow Y$
we get induced maps $(\pi_X)_*, (\pi_Y)_*$.

Define $\Phi : \pi_1(X\times Y, x_0\times y_0) \longrightarrow \pi_1(X, x_0)\times\pi_1(Y, y_0)$

$$\Phi([f]) = (\pi_X)_*([f]) \times (\pi_Y)_*([f])$$

This is a group homomorphism.

Claim 1: Φ is injective:

If $\Phi([f]) = [e_{x_0}]\times[e_{y_0}]$, we want $[f] = [e_{x_0\times y_0}]$.

Let $h = \pi_X(f)$, $g = \pi_Y(f)$. The above condition
$\Rightarrow$ $\exists$ a homotopy H from h to e_{x_0},
and a homotopy G from g to e_{y_0}.
Then (H, G) is a homotopy from $\underset{h\times g}{\overset{\shortparallel}{f}}$ to $e_{x_0}\times e_{y_0}$.

Claim 2: Φ is surjective.

For any pair $[h]\times[g] \in \pi_1(X, x_0)\times\pi_1(Y, y_0)$ we note that $(h, g) = f$ is a loop in $X\times Y$ based at $x_0\times y_0$ with the property that

$$\Phi([f]) = [h]\times[g].$$

Exercise: $\pi_1(\mathbb{R}^n\setminus\{a_1, \dots, a_k\}, x_0) = \{[e_{x_0}]\}$ $\forall\, n\geq 3$.

ie, $\mathbb{R}^n\setminus$ finitely many points is si connected when $n\geq 3$.

Def: "Projective plane" $\mathbb{P}^2 = S^2/\sim$ where $x\sim -x$ $\forall\, x\in S^2$.

Theorem: $\mathbb{P}^2$ is compact, and the quotient map $p : S^2 \longrightarrow \mathbb{P}^2$ is a covering.

Corollary: $\pi_1(\mathbb{P}^2, [x_0]) \cong \mathbb{Z}/2\mathbb{Z}$

Proof: Note: $p^{-1}([x_0]) = \{x_0, -x_0\}$.

Any simple path α on S^2 from x_0 to $-x_0$ projects to a non-trivial loop f on $\mathbb{P}^2$.

However, $f * f$ is the projection of a loop γ on S^2 based at x_0. We will show later on that $\pi_1(S^2, x_0)$ is trivial. $\therefore$ $[\gamma] = [e_{x_0}]$, and so $[f*f] = p_*([\gamma]) = [e_{[x_0]}]$.

Proof of Theorem: Note that $\forall\, [y_0]\in\mathbb{P}$, $\exists$ a neighborhood $V\subset\mathbb{P}^2$ s.t. $p^{-1}(V)$ is of the form $U\cup(-U)$ where $U\cap(-U) = \emptyset$.

Eg: y_0 & $-y_0$ determine two hemispheres of S^2.
Take $U\ni y_0$ to be strictly within the hemisphere containing y_0. Then $U\cap(-U) = \emptyset$ and we can simply take $V = p(U)$.

Seifert - Van Kampen Theorem for S^n

Recall that $S^n = \partial B^n = \{x\in\mathbb{R}^{n+1} : d(x, 0) = 1\}$.

Theorem: Suppose $X = U\cup V$ where U, V are open in X.
Say $U\cap V$ is path connected and $x_0\in U\cap V$.
Let $i : U \hookrightarrow X$ and $j : V\hookrightarrow X$ be the standard inclusions. Then the images of

$i_* : \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0)$

$j_* : \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$

generate $\pi_1(X, x_0)$.

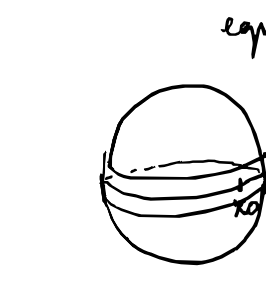
(ie, any loop in X based at x_0 is homotopic to the product of loops in U and V based at x_0)

Corollary 1: with the assumptions of the above theorem, If U, V are simply connected, then so is X.

Proof: $\pi_1(U, x_0)$ and $\pi_1(V, y_0)$ are trivial, and so are their images under i_* and j_*. Therefore $\pi_1(X, x_0)$ is trivial.

Corollary 2: If $n\geq 2$, the n-sphere S^n is simply connected.

Proof: Proof for $n=2$, $n>2$ is similar.
Take a base point x_0 on the equator of S^n, and let A be a small band around the equator which is open in S^n.



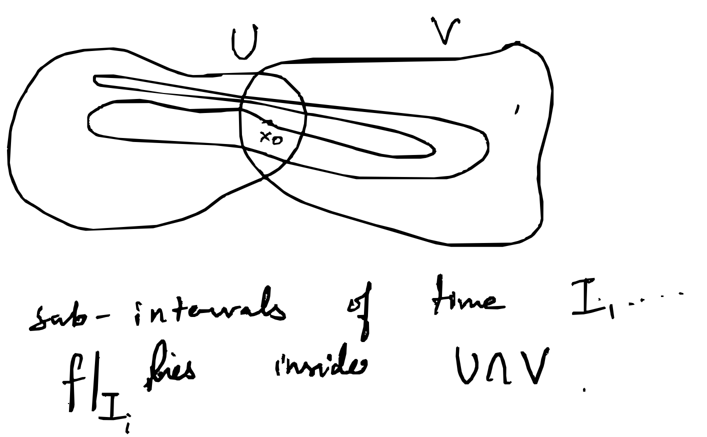
Let U = upper hemisphere $\cup A$,
V = lower hemisphere $\cup A$.

Then $U\cap V = A$ is path connected, and contains x_0. Note that U & V are both homeomorphic to disks, and are therefore simply connected. By the previous corollary, S^n is simply connected.

Proof of Theorem: Let $[f]\in\pi_1(X, x_0)$ so that f is not homotopic to a loop entirely inside U or V.

Claim 1: $\exists$ a subdivision $0 = a_0 < a_1 < \dots < a_n = 1$ of $[0,1]$ s.t. $f(a_i)\in U\cap V$ and $f|_{[a_i, a_{i+1}]}\subset U$ or V $\forall\, i$.

Proof of claim 1:



List all the sub-intervals of time $I_1, \dots, I_n$ in $[0,1]$ where $f|_{I_i}$ lies inside $U\cap V$.

Claim 2: Claim 1 $\Rightarrow$ Theorem.

Proof of claim 2.